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Explicit Polynomial Generators for the Ring of Quasisymmetric Functions over the Integers

Générateurs Explicites pour les Fonctions Quasisymétriques sur les Entiers Rationels

Michiel Hazewinkel

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Abstract In (Hazewinkel in Adv. Math. 164:283–300, 2001, and CWI preprint, 2001) it
been proved that the ring of quasisymmetric functions over the integers is free polynomi
This is a matter that has been of great interest since 1972; for instance because of the role t
statement plays in a classification theory for noncommutative formal groups that has been
development since then, see (Ditters in Invent. Math. 17:1–20, 1972; in Scholtens' The
Free Univ. of Amsterdam, 1996) and the references in the latter. Meanwhile quasisymmet
functions have found many more applications (see Gel'fand et al. in Adv. Math. 112:21
348, 1995). However, the proofs of the author in the aforementioned papers do not g
explicit polynomial generators for *QSymm* over the integers. In this note I give a (rea
quite simple) set of polynomial generators for *QSymm* over the integers.

Résumé Dans (Hazewinkel dans Adv. Math. 164 :283–300, 2001, et CWI preprint, 20
il a été démontré que l'anneau de fonctions quasisymétriques est polynomialement libre
l'anneau de base **Z**. C'est là une question importante etudiée depuis 1972 ; par exem
cet énoncé joue un rôle important dans la théorie de la classification des groupes form
noncommutatifs, voir (Ditters dans Invent. Math. 17 :1–20, 1972 ; Scholtens dans The
Free Univ. of Amsterdam, 1996 et les références données). Entretemps, les fonctions q
sisymétriques ont reçu beaucoup d'applications (voir Gel'fand et al. dans Adv. Math. 11
218–348, 1995). Par contre les démonstrations données par l'auteur dans les articles cité p
haut ne fournissent pas des générateurs polynomiaux explicites pour *QSymm* sur l'ann
des entiers rationels. Dans cette Note nous présentons un ensemble (vraiment très sim
de générateurs polynomiaux pour *QSymm* sur **Z**.

Keywords Quasisymmetric function · Symmetric function · Plethysm · λ-ring · Lambd
ring · Frobenius operator · Adams operator

M. Hazewinkel (✉)
Burg. s'Jacob laan 18, 1401BR Bussum, The Netherlands
e-mail: michhaz@xs4all.nl

M. Hazewinkel
e-mail: mich@cwi.nl

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Version Française Abrégée

Le but de cette Note est de donner des générateurs explicites pur l'anneau des fonctions quasisymétriques sur l'anneau des entiers rationels **Z**. Alors, soit $\mathbf{Z}[x_1, x_2, \ldots]$ l'anneau des polynômes a coefficients entiers en $x_1, x_2, \ldots$ et, comme d'habitude, soient

$$Symm \subset QSymm \subset \mathbf{Z}[x_1, x_2, \ldots]$$

les sous-anneaux des fonctions symétriques et quasisymétriques. Si $\alpha = [a_1, \ldots, a_m]$ est une composition de n, ça veut dire une suite d'entiers naturels $a_l, a_2, \ldots, a_m$ telle que $\sum_i a_i = n$, la fonction quasisymétrique monomiale associée est

$$\alpha = \sum_{j_1 < \cdots < j_m} x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_m}^{a_m}$$

Ces fonctions quasisymetriques monomiales forment une base de *QSymm* (comme groupe Abélien libre).

Il y a une structure de λ-anneau sur $\mathbf{Z}[x_1, x_2, \ldots]$, donnée par

$$\lambda_i(x_j) = \begin{cases} x_j & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases} \qquad j = 1, 2, \ldots$$

Les sous-anneaux *Symm* et *QSymm* sommes stable sous ces operations.

Un mot de Lyndon $\alpha = [a_1, \ldots, a_m]$ est dite élémentaire si $(a_1, a_2, \ldots, a_m) = 1$. Soit *eLYN* l'ensemble des mots de Lyndon élémentaires.

Théorème Les $e_n(\alpha) = \lambda^n(\alpha)$, $\alpha \in eLYN$, $n = l, 2, \ldots$, forment une base polynomiale libre pour *QSymm* sur **Z**.

Si $\alpha = [1]$, $e_n(\alpha) = e_n$, la fonction symétrique élémentaire de degrée n, ce qui peut expliquer la notation.

Les opérateurs de Adams associés à la structure de λ-anneau sur *QSymm* sont données sur les fonctions quasisymétriques monomiales par

$$\mathbf{f}_n([a_l, \ldots, a_m]) = [na_l, \ldots, na_m]$$

Il y a des formules bien connues qui relient les λ_i et les $\mathbf{f}_i$, ce qui permet d'écrire des expressions explicites pour les $e_n(\alpha)$.

1 Introduction

As indicated in the abstract a somewhat important problem is the finding of explicit free polynomial generators for the ring of quasisymmetric functions over the integers. A seminal inspirational formula for this was (and is) the following observation

$$\exp\left(\sum_{n=1}^{\infty} n^{-1}[na_1, na_2, \ldots, na_m]t^n\right) \in QSymm[[t]] \tag{1.1}$$

It will not be very clear from what follows just what this formula has to do with the actu proof below. (But see Remark 3.2.) The fact that this formula is actually of central impc tance is quite effectively hidden in the use of plethysms as employed in Sect. 3.

A much more detailed paper explaining this and much more is in preparation.

2 Lambda Ring Structure on *QSymm*

Consider the rings of symmetric functions and quasisymmetric functions in infinitely ma variables $x_1, x_2, \ldots$ over the integers

$$Symm = \mathbf{Z}[e_l, e_2, \ldots] \subset QSymm \subset \mathbf{Z}[x_l, x_2, \ldots] \qquad (2.$$

Here the e_i are the elementary symmetric functions in the x_j. For some details on qua symmetric functions and definitions of various concepts used below see [4–7]. There is well-known λ-ring structure on $\mathbf{Z}[x_1, x_2, \ldots]$ given by

$$\lambda_i(x_j) = \begin{cases} x_j & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases} \qquad j = 1, 2, \ldots \qquad (2.$$

The associated Adams operators, determined by the formula

$$t\frac{d}{dt}\log\lambda_t(a) = \sum_{n=1}^{\infty}(-1)^n \mathbf{f}_n(a)t^n \qquad (2$$

where

$$\lambda_t(a) = l + \sum_{n=1}^{\infty}\lambda_n(a)t_n \qquad (2$$

are the ring endomorphisms

$$\mathbf{f}_n : x_j \mapsto x_j^n \qquad (2$$

There are well-known determinantal relations between the λ_n and the $\mathbf{f}_n$ as follows

$$n!\lambda_n(a) = \det\begin{pmatrix} \mathbf{f}_1(a) & 1 & 0 & \ldots & 0 \\ \mathbf{f}_2(a) & \mathbf{f}_1(a) & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \mathbf{f}_{n-1}(a) & \mathbf{f}_{n-2}(a) & \ldots & \mathbf{f}_1(a) & n-1 \\ \mathbf{f}_n(a) & \mathbf{f}_{n-1}(a) & \ldots & \mathbf{f}_2(a) & \mathbf{f}_1(a) \end{pmatrix} \qquad (2$$

$$\mathbf{f}_n(a) = \det \begin{pmatrix} \lambda_1(a) & 1 & 0 & \dots & 0 \\ \lambda_2(a) & \lambda_1(a) & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ (n-1)\lambda_{n-1}(a) & \lambda_{n-2}(a) & \dots & \lambda_1(a) & 1 \\ n\lambda_n(a) & \lambda_{n-1}(a) & \dots & \lambda_2(a) & \lambda_1(a) \end{pmatrix} \tag{2.7}$$

(which come from the Newton relations between the elementary symmetric functions and the power sum symmetric functions).

It follows that the subrings *Symm* and *QSymm* are stable under the λ_n and $\mathbf{f}_n$ because $\lambda_n(QSymm \otimes_{\mathbf{Z}} \mathbf{Q}) \subset QSymm \otimes_{\mathbf{Z}} \mathbf{Q}$ by (2.6) and $(QSymm \otimes_{\mathbf{Z}} \mathbf{Q}) \cap \mathbf{Z}[x_1, x_2, \dots] = QSymm$, and similarly for *Symm*. It follows that *QSymm* and *Symm* have induced λ-ring structures.

A composition $\alpha = [a_l, a_2, \dots, a_m]$, $a_i \in \mathbf{N} = \{1, 2, \dots\}$ defines a monomial quasisymmetric function also denoted by α

$$\alpha = \sum_{i_1 < i_2 < \dots < i_m} x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_m}^{a_m} \tag{2.8}$$

The empty composition corresponds to the quasisymmetric function 1. The monomial quasisymmetric functions form a basis for *QSymm* as an Abelian group.

Lemma 2.1 $\mathbf{f}_n([a_l, a_2, \dots, a_m]) = [na_l, na_2, \dots, na_m]$.

This follows immediately from (2.8) and (2.5).

Give a composition $\alpha = [a_1, a_2, \dots, a_m]$ weight $\mathrm{wt}(\alpha) = a_1 + a_2 + \dots + a_m$ and length $\lg(\alpha) = m$. The wll-ordering on compositions is defined by "weight first, then length, then lexicographic". Thus for instance

$$[5] >_{\text{wll}} [1, 1, 2] >_{\text{wll}} [2, 2] >_{\text{wll}} [1, 3]$$

Lemma 2.2 *If α is a Lyndon word (= Lyndon composition)*

$$\lambda_n(\alpha) = \alpha^{*n} + (\textit{wll-smaller than } \alpha^{*n}) \tag{2.9}$$

where $*$ denotes concatenation (of compositions) and (wll–smaller than α^{*n}) stands for a $\mathbf{Z}$-linear combination of monomial quasisymmetric functions that are wll-smaller than α^{*n}.

This follows immediately from formula (2.6) and Lemma 2.1. Indeed, expanding the determinant (2.6) we see that

$$n!\lambda_n(a) = \alpha^n + (\text{monomials of length} \le (n-1) \text{ in the } \mathbf{f}_i(\alpha))$$

All monomials occurring in $\lambda_n(\alpha)$ are of equal weight $n\mathrm{wt}(\alpha)$. By the formula above the longest ones come from the power α^n and are of length $n\lg(\alpha)$. Because a is Lyndon the lexicographic largest term of these is α^{*n} and it occurs with coefficient $n!$.

For any λ-ring R there is an associated mapping

$$Symm \times R \longrightarrow R, \quad (\varphi, a) \mapsto \varphi(\lambda_1(a), \lambda_2(a), \dots, \lambda_n(a), \dots) \tag{2.10}$$

I.e. write $\varphi \in Symm$ as a polynomial in the elementary symmetric functions $e_1, e_2, \dots$ ar then substitute $\lambda_i(a)$ for e_i, $i = 1, 2, \dots$. For fixed $a \in R$ this is obviously a homomorphis of rings $Symm \longrightarrow R$. We shall often simply write $\varphi(a)$ for $\varphi(\lambda_1(a), \lambda_2(a), \dots, \lambda_n(a), \dots$ Another way to see (2.10) is to observe that for fixed $a \in R$ $(\varphi, a) \mapsto \varphi(\lambda_1(a), \lambda_2(a), \dots)$ $\varphi(a)$ is the unique homomorphism of λ-rings that takes e_i into a. (*Symm* is the free λ-rir on one generator, see also [7, 8].) Note that

$$e_n(\alpha) = \lambda_n(\alpha), \qquad p_n(\alpha) = \mathbf{f}_n(\alpha) = [na_l, na_2, \dots, na_m] \qquad (2.1$$

The first formula of (2.11) is by definition and the second follows from (2.7) because tl relations between the e_n and p_n are precisely the same as between the $\lambda_n(a)$ and the $\mathbf{f}_n($ (see (2.3)).

Lemma 2.3 *For any $\varphi, \psi \in Symm$ and $a \in R$, where R is a λ-ring*

$$\varphi(\psi(a)) = (\varphi \circ \psi)(a) \qquad (2.1$$

where $\varphi \circ \psi$ is the (outer) plethysm of $\varphi, \psi \in Symm$.

This is well known, see [8], p. 134, Remark 1. For our purposes here it does not matt just how plethysm is defined. The only thing needed is that there is some element $\varphi \circ \psi$ *Symm* such that (2.12) holds.

3 Explicit Polynomial Generators for *QSymm*

Let *LYN* denote the set of Lyndon words and let *eLYN* be the set of elementary (or reduce Lyndon words, i.e. the set of those Lyndon words $\alpha = [a_1, a_2, \dots, a_m]$ for which $g(\alpha)$ $\gcd\{a_l, a_2, \dots, a_m\} = l$.

Theorem 3.1 *The $e_n(\alpha)$, $\alpha \in eLYN$, $n \in \mathbf{N}$ form a free set of polynomial generators over t integers for the ring of quasisymmetric functions QSymm.*

Proof The difficult part is to prove generation, i.e. that every basis element β (in the Abeli group sense) of *QSymm* can be written as a polynomial in the $e_n(\alpha)$, $\alpha \in eLYN$, $n \in \mathbf{N}$. T rest follows by the same counting argument that was used in, e.g., [5]. So let us pro generation.

To start with, let $\beta = [b_1, b_2, \dots, b_m]$ be a Lyndon composition. Then taking $\alpha = \beta_{\text{red}}$ $[g(\beta)^{-1}b_1, g(\beta)^{-1}b_2, \dots, g(\beta)^{-1}b_m]$, and $n = g(\beta)$ we have, using (2.11), $\beta = p_n(\alpha)$ whi is a polynomial in the $e_n(\alpha)$, and thus $\beta \in R$.

We now proceed by induction on the wll-ordering. The case of weight 1 is trivial. F each separate weight the induction starts because of what has just been said because co positions of length 1 are Lyndon.

So let β be a composition of weight ≥ 2 and length ≥ 2. By the Chen-Fox-Lynd concatenation factorization theorem [1]

$$\beta = \beta_1^{*r_1} * \beta_2^{*r_2} * \dots * \beta_k^{*r_k}, \quad \beta_i \in LYN, \quad \beta_1 >_{\text{lex}} \beta_2 >_{\text{lex}} \dots >_{\text{lex}} \beta_k \qquad (3$$

where, as before, the $*$ denotes concatenation and $\beta >_{\text{lex}} \beta'$ means that β is lexicographica strictly larger than β'.

If $k \geq 2$, take $\beta' = \beta_1^{*r_1}$ and for β'' the corresponding tail of β so that $\beta = \beta' * \beta''$. Then

$$\beta'\beta'' = \beta' * \beta'' + (\text{wll-smaller than } \beta) = \alpha + (\text{wll-smaller than } \beta)$$

and with induction it follows that $\beta \in R$.

There remains the case that $k = 1$ in the CFL-factorization (3.1). In this case take $\alpha = (\beta_1)_{\mathrm{red}}$ and observe that by Lemma 2.1 and (2.11)

$$\beta = e_{r_1}(p_{g(\beta_1)}(\alpha)) + (\text{wll-smaller than } \beta) \tag{3.2}$$

On the other hand, by Lemma 2.3

$$e_{r_1}(p_{g(\beta_1)}(\alpha)) = (e_{r_1} \circ p_{g(\beta_1)})(\alpha) \tag{3.3}$$

Here $e_{r_1} \circ p_{g(\beta_1)}$ is some polynomial with integer coefficients in the e_j, and hence $(e_{r_1} \circ p_{g(\beta_1)})(\alpha)$ is a polynomial with integer coefficients in the $e_j(\alpha)$. With induction this finishes the proof. □

Remark 3.2 In terms of the $e_n(\alpha)$, for any composition $\alpha = [a_l, a_2, \ldots, a_m]$ the exponential from the introduction is equal to

$$\exp\left(\sum_{n=1}^{\infty}(-1)^{n-1}n^{-1}[na_1, na_2, \ldots, na_m]t^n\right) = 1 - e_1(\alpha)t + e_2(\alpha)t^2 - e_3(\alpha)t^3 + \cdots$$

Acknowledgements Define the quasisymmetric functions $a_n(\alpha)$ by the triangular relation

$$\prod_n (1 - a_n(\alpha)t^n) = 1 - e_1(\alpha)t + e_2(\alpha)t^2 - e_3(\alpha)t^3 + \cdots$$

Of course the relations between $a_n(\alpha)$ and the $e_n(\alpha)$ are given by triangular matrices with ±1 on the diagonal. So if one is a set of generators so is the other. In Spring 2002. E.J. Ditters wrote me a letter stating that the $a_n(\alpha)$ are a free polynomial set of generators of *QSymm*. The proof rests on observing that these elements when evaluated on the explicit basis of Prim(*NSymm*) from [6] give a triangular matrix with ±1 on the diagonal. Immediately afterwards I gave a proof that did not use duality but still used the abstract theorem that *QSymm* was free. Some months later I found the proof that is in this note. Thus, historically speaking, this is the fifth proof that *QSymm* is freely generated and the third one that also provides an explicit set of generators.

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